

Seeking the balance: patching double and exceptional field theories

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ABSTRACT: We investigate the patching of double and exceptional field theories. In double field theory the patching conditions imposed on the spacetime after solving the strong section condition imply that the 3-form field strength H is exact. A similar conclusion can be reached for the form field strengths of exceptional field theories after some plausible assumptions are made on the relation between the transition functions of the additional coordinates and the patching data of the form field strengths. We illustrate the issues that arise, and explore several alternative options which include the introduction of C-folds and of the topological geometrisation condition.

KEYWORDS: Double Field Theory, Patching

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1 Introduction

It has been known for sometime that given a spacetime M and a 2-form gauge field, F , $dF = 0$, one can introduce an additional coordinate, a charge coordinate, and together with the Dirac quantisation condition, one can construct a $U(1)$ bundle M_F over the spacetime M . This bundle is a new manifold associated to the Maxwell field and the transition functions of the fibre are related to the patching conditions of the 1-form gauge potential of F . This construction is a manifestation of the isomorphism of $U(1)$ bundles over M with the cohomology classes in $H^2(M, \mathbb{Z})$ and underpins Kaluza-Klein theory. In the latter, the $U(1)$ gauge potential is a component of the metric in higher dimensions. This is sometimes referred as the geometrisation¹ of the $U(1)$ field. A feature of the construction is that the

¹Later, we shall introduce a related notion referred to as “topological geometrisation”.

charge of the $U(1)$ field is replaced by information stored in the patching conditions of M_F .

Similar suggestions have emerged in the context of string theory and M-theory following the early works of [1–3]. These include double field theory (DFT) of [4–7] applied to string theory, the E_{11} [8, 9], and the exceptional field theory (EFT) [10–14] proposals, see also reviews [15–17] and references within. There are several reasons for this. One is to find a geometric realization of string and M-theory dualities and to describe duality covariant theories. Another is to explore the idea that string and M-theory dualities emerge as symmetries of the 10- and 11-dimensional theories rather than just their toroidal compactifications following the early work of [18]. The constructions are broadly based on a similar technology to the 2-form $U(1)$ gauge field described above but now the metric in higher dimensions is replaced by a generalized metric which includes the form gauge potentials of string theory and M-theory, and the introduction of suitable new coordinates.

A related construction is that of generalized geometry [19, 20]. In generalized geometry no new coordinates are introduced in addition to those of spacetime. Instead the tangent space of the spacetime is replaced by a vector bundle E which is an extension of TM equipped with an appropriate bracket. Such an approach has been used to explore some geometric properties of supergravities associated with strings and M-theory [21–24].

Although much work has been done to understand the geometry that underpins DFT and EFT several questions remain. One question is to unravel topological and differential structures of double and exceptional spaces, and another related question is to understand how information about string and brane charges is stored in their topology. In the context of the formalism developed so far, it is not possible to answer these questions because most of the computations have been made using infinitesimal transformations generated by generalized Lie derivatives. However for DFT a set of finite transformations have been proposed [17, 25] by integrating the infinitesimal transformations that have been known before, and have been explored as transition functions for double spaces in [26]. Infinitesimal transformations for the coordinates of exceptional spaces underlying EFT have been proposed in eg [10, 11, 13, 27, 28].

In this paper, we shall review some of the properties of patching closed form field strengths on a spacetime and illustrate the issues involved. Then we shall demonstrate that the transformations of [17, 25] that solve the strong section condition after interpreting them as patching conditions following [26] imply that the NS-NS 3-form field strength H which arises in string theory is an exact 3-form.

In addition, we explore the double space of T^3 with H-flux background of [17] from the patching conditions point of view. We find that without additional assumptions on the patching conditions, the construction is inconsistent as it requires that H is exact. However, we also show that it is possible to modify the patching conditions at triple overlaps in such a way that consistency is restored. But such a modification is dependent on the existence of a particular atlas on T^3 , ie it is not general covariant, and so it cannot be adapted to other backgrounds.

We shall further argue that a similar conclusion can be reached in the context of EFT provided that the patching conditions of closed form field strengths are related in a linear way to the transition functions of the additional coordinates of the exceptional spaces

and the combinatorial law follows the usual rules of tensor calculus. The latter point will be illustrated in the $U(1)$ paradigm reviewed in the next section. Concerns about the consistency of patching EFT have also been raised in [31] using the Cartan theorem [32].

We shall explore several alternative ways to reconcile the transition functions of the double and exceptional spaces with the patching conditions of the closed form field strengths on the spacetime. We shall see that there are examples described in appendix A, where this can be done at a cost. In particular, these constructions do not exhibit some key properties of the $U(1)$ paradigm. Nevertheless, they illustrate some of the issues involved and provide a local model of a consistent construction.

In the conclusions, we propose a general scheme with a minimum number of requirements that should be followed in order to construct double and exceptional spaces which exhibit the key properties of the $U(1)$ paradigm and allow for a realization of duality groups. To distinguish them from previous constructions, we shall call these new spaces “C-folds”, ie manifolds with charge coordinates. One of the key requirements that we propose is the “topological geometrisation condition”. This states that the pull back on the C-fold of the closed form field strengths on a spacetime must be exact. One of the consequences of this condition is that the fibres of the C-fold must have a non-trivial topology. We propose several constructions for C-folds based on K-theory and homotopy theory.

This paper has been organized as follows. In section 2, we describe the $U(1)$ field paradigm. In section 3, we show that for DFT the patching conditions allow for only exact form field strengths. In section 4, we revisit the construction of the double space of T^3 with H -flux background. In section 5, we show under certain assumptions a similar statement for the EFT. In section 6, we give our conclusions and explore some future directions. In appendix A, we give an example of an alternative construction for double and exceptional geometries consistent with closed form field strengths.

2 A review of $U(1)$ field paradigm

Before we proceed to investigate how to extend a spacetime with additional coordinates associated to a general closed k -form, it is instructive to revisit the construction for $U(1)$ fields which is well known.

To begin, suppose M is a n -dimensional manifold equipped with a good cover $\{U_\alpha\}_{\alpha \in I}$, see eg [34], and a closed 2-form ω^2 . On each open set U_α , the Poincaré theorem implies that there are 1-form potentials C_α^1 such that

$$\omega_\alpha^2 = dC_\alpha^1 . \quad (2.1)$$

Using repeatedly the Poincaré lemma at double and triple overlaps $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$, one has that

$$C_\alpha^1 - C_\beta^1 = da_{\alpha\beta}^0 , \quad a_{\alpha\beta}^0 + a_{\beta\gamma}^0 + a_{\gamma\alpha}^0 = n_{\alpha\beta\gamma} , \quad (2.2)$$

where $n_{\alpha\beta\gamma}$ are constants. Now if ω^2 represents a class in $H^2(M, \mathbb{Z})$, then $n_{\alpha\beta\gamma} \in 2\pi\mathbb{Z}$ are integers.

The geometrisation of the $U(1)$ field proceeds as follows. Starting from the good cover $\{U_\alpha\}_{\alpha \in I}$ of M , one introduces a new space with charts $U_\alpha \times \mathbb{R}$, and transition functions

$$x_\alpha = f_{\alpha\beta}(x_\beta) , \quad \theta_\alpha = \theta_\beta + a_{\alpha\beta}^0 , \quad (2.3)$$

where x_α and $f_{\alpha\beta}$ are the coordinates and transition functions of M and θ_α is an additional coordinate. At first sight it appears that the second transition function is consistent with (2.2) at triple overlaps iff $n_{\alpha\beta\gamma} = 0$. As we shall prove later, this implies that ω^2 is exact. However, for any closed ω^2 which represents a class in $H^2(M, \mathbb{Z})$, the transition functions are consistent at triple overlaps by taking the addition in the second transition function in (2.3) to be over mod $2\pi\mathbb{Z}$, ie the new transition functions are

$$x_\alpha = f_{\alpha\beta}(x_\beta) , \quad (\theta_\alpha - \theta_\beta - a_{\alpha\beta}^0) = 0 \bmod 2\pi\mathbb{Z} . \quad (2.4)$$

In such a case no further condition arises as the sum over a^0 's in (2.2) at triple overlaps vanishes mod $2\pi\mathbb{Z}$.

The effect of this construction is to define a circle bundle M_{ω^2} over M with transition functions $\phi_{\alpha\beta} = \exp(ia_{\alpha\beta}^0)$. Some of the features and consequences of this construction are as follows.

- (i) An integral part of the construction is the modification of the combinatorial law that it is used to describe the transition functions of the additional coordinate. This is related to the requirement that ω^2 represents a class in $H^2(M, \mathbb{Z})$.
- (ii) The topological structure of M_{ω^2} is completely determined by the class $[\omega^2]$ of ω^2 in $H^2(M, \mathbb{Z})$, ie $M_{\omega^2} = M_{[\omega^2]}$, and vice versa, as there is a 1-1 correspondence between circle bundles and elements of $H^2(M, \mathbb{Z})$.
- (iii) The tangent bundle TM_{ω^2} of M_{ω^2} is an extension of TM with respect to a trivial real line bundle L , ie

$$0 \rightarrow L \rightarrow TM_{\omega^2} \rightarrow \pi^*TM \rightarrow 0 \quad (2.5)$$

where $\pi : M_{\omega^2} \rightarrow M$ is a projection.

- (iv) Another feature of the construction is that $\pi^*\omega^2$ is an exact form on M_{ω^2} as

$$\pi^*\omega^2 = -d(d\theta - \pi^*C^1) \quad (2.6)$$

and $(d\theta - \pi^*C^1)$ is a globally defined 1-form on M_{ω^2} .

The last property can be seen as the “topological geometrisation” of ω^2 . On M , ω^2 has charges which are given by the periods

$$n_i = \int_{B_i} \omega^2 \quad (2.7)$$

where (B_i) is a basis in $H_2(M, \mathbb{Z})$. Since $\pi^*\omega^2$ is exact on M_{ω^2} , all the periods of $\pi^*\omega^2$ on M_{ω^2} vanish. Instead all the information carried by the periods n_i has been replaced by the topology of M_{ω^2} .

The above construction can be generalized to include more than one $U(1)$ field strengths leading to toric fibrations over M . In addition this can be generalized to non-abelian gauge fields which in turn leads to principal bundles with fibre the gauge group. In both cases all the properties mentioned above, with some minor modifications, apply to the more general set up.

3 Patching 3-forms and double coordinates

3.1 Transition functions of closed 3-forms

Before, we proceed to the patching of k -forms and the introduction of new coordinates, it is instructive to explain the patching of closed 3-forms ω^3 . For this let M be a n -dimensional manifold² with a good cover $\{U_\alpha\}_{\alpha \in I}$ and a partition of unity $\{\rho_\alpha\}_{\alpha \in I}$ subordinate to $\{U_\alpha\}_{\alpha \in I}$. This means that M admits functions $\rho_\alpha \geq 0$ with support in U_α such that at every point $x \in M$

$$\sum_{\alpha} \rho_\alpha = 1, \quad (3.1)$$

where the sum is taken over a finite collection. For a discussion on partitions of unity and the different kinds that exist see [34]. Here we shall use those partitions of unity that have the same index set I as that of the good cover and so they do not have necessarily compact support.

To continue, take ω^3 to be a closed 3-form on M . Using the Poincaré lemma at each open set U_α , we can write ω^3 in terms of a gauge potential C as

$$\omega^3_\alpha = dC_\alpha^2. \quad (3.2)$$

Using repeatedly the Poincaré lemma at the double, $U_\alpha \cap U_\beta$, triple, $U_\alpha \cap U_\beta \cap U_\gamma$, and quadruple, $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$, overlaps, we have

$$C_\alpha^2 = C_\beta^2 + da_{\alpha\beta}^1, \quad a_{\alpha\beta}^1 + a_{\beta\gamma}^1 + a_{\gamma\alpha}^1 = da_{\alpha\beta\gamma}^0, \quad a_{\alpha\beta\gamma}^0 - a_{\delta\alpha\beta}^0 + a_{\gamma\delta\alpha}^0 - a_{\beta\gamma\delta}^0 = n_{\alpha\beta\gamma\delta} \quad (3.3)$$

where $n_{\alpha\beta\gamma\delta}$ are constants. Note that the patching data $a_{\alpha\beta}^1, a_{\alpha\beta\gamma}^0, n_{\alpha\beta\gamma\delta}$ are skew-symmetric in the interchange of any two of the open set labels, ie $a_{\beta\gamma}^1 = -a_{\gamma\beta}^1$ and similarly for the rest. Moreover $n_{\alpha\beta\gamma\delta}$ are restricted to be multiples of integers if ω^k represents a class in $H^3(M, \mathbb{Z})$. In particular, the patching condition of 2-form gauge potential in local coordinates reads

$$(C_\alpha^2)_{ij} = \frac{\partial x_\beta^k}{\partial x_\alpha^i} \frac{\partial x_\beta^l}{\partial x_\alpha^j} (C_\beta^2)_{kl} + \frac{\partial}{\partial x_\alpha^i} a_{\alpha\beta,j}^1 - \frac{\partial}{\partial x_\alpha^j} a_{\alpha\beta,i}^1$$

²In what follows, the signature of M is not essential and the results apply to both Riemannian and Lorentzian manifolds.

$$= \frac{\partial x_\beta^k}{\partial x_\alpha^i} \frac{\partial x_\beta^l}{\partial x_\alpha^j} \left((C_\beta^2)_{kl} + \frac{\partial}{\partial x_\beta^k} a_{\alpha\beta,l}^1 - \frac{\partial}{\partial x_\beta^l} a_{\alpha\beta,k}^1 \right) . \quad (3.4)$$

Observe that the choice of gauge potentials is not unique. In particular, they are defined up to the gauge transformation

$$C_\alpha^2 \rightarrow C_\alpha^2 + d\chi_\alpha^1 . \quad (3.5)$$

Similarly, the patching data $a_{\beta\gamma}^1$ of ω^3 at double overlaps are not uniquely defined either. In particular, they are defined up to a gauge transformation

$$a_{\alpha\beta}^1 \rightarrow a_{\alpha\beta}^1 - \chi_\alpha^1 + \chi_\beta^1 + d\psi_{\alpha\beta}^0 , \quad (3.6)$$

for a 0-form $\psi_{\alpha\beta}^0$ defined on double overlaps. This is the only ambiguity that one has in determining the patching conditions on double overlaps of ω^3 . Anything else is inconsistent with the identification of ω^3 as a closed 3-form on M .

3.2 Transition functions and exact 3-forms

Before we proceed to compare the above patching conditions of closed 3-forms with those that arise in DFT, we shall prove a technical lemma which arises in the context of Čech-de Rham theory. In particular, if there is a choice of patching data, up to (3.6) gauge transformations, such that $a_{\alpha\beta}^1$ on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ satisfies the cocycle³ condition

$$a_{\alpha\beta}^1 + a_{\beta\gamma}^1 + a_{\gamma\alpha}^1 = 0 , \quad (3.7)$$

then ω^3 is exact.

For this suffices to show that $\omega^3 = d\tilde{C}^2$, where \tilde{C}^2 is a 2-form on M , ie $\tilde{C}_\alpha = \tilde{C}_\beta$. Indeed define

$$\tilde{C}_\alpha^2 = C_\alpha^2 + \sum_\gamma d(\rho_\gamma a_{\gamma\alpha}^1) . \quad (3.8)$$

Observe that $d\tilde{C}_\alpha = dC_\alpha = \omega^3$ and

$$\begin{aligned} \tilde{C}_\alpha - \tilde{C}_\beta &= C_\alpha^2 - C_\beta^2 + \sum_\gamma d(\rho_\gamma a_{\gamma\alpha}^1) - \sum_\gamma d(\rho_\gamma a_{\gamma\beta}^1) \\ &= da_{\alpha\beta}^1 - \sum_\gamma d(\rho_\gamma a_{\alpha\beta}^1) = da_{\alpha\beta}^1 - da_{\alpha\beta}^1 = 0 , \end{aligned} \quad (3.9)$$

where to establish the last two equalities we have used (3.7) and (3.1), respectively. Thus if (3.7) holds, ω^3 is exact and represents the trivial class in $H^3(M, \mathbb{R})$.

³This is in the sense of Čech cohomology, see [34].

3.3 Patching DFT

After integrating the infinitesimal transformations generated by generalized Lie derivatives, generalized finite tensor transformations have been proposed for DFT⁴ in [17, 25] and further explored as transition functions in [26]. According to these generalized tensor transformations, 1-forms are transformed as

$$\omega'_N = F_N^M \omega_M , \quad (3.10)$$

where

$$F_N^M = \frac{1}{2} \left(\frac{\partial X^P}{\partial X'^N} \frac{\partial X'_P}{\partial X^M} + \frac{\partial X'_N}{\partial X^P} \frac{\partial X^M}{\partial X'^P} \right) , \quad (3.11)$$

and where X^M are the coordinates of the double space and indices are raised and lowered with the split metric η_{MN} and $X'^M = X'^M(X^N)$. All the fields and coordinate transformations satisfy the strong section condition.

Viewing the above transformations as patching conditions and putting them into the language of the previous section, we have

$$X_\alpha^M = X_{\alpha\beta}^M(X_\beta^N) , \quad (\omega_\alpha)_N = (F_{\alpha\beta})_N^M (\omega_\beta)_M , \quad (3.12)$$

where

$$(F_{\alpha\beta})_N^M = \frac{1}{2} \left(\frac{\partial X_\beta^P}{\partial X_\alpha^N} \frac{\partial X_{\alpha P}}{\partial X_{\beta M}} + \frac{\partial X_{\alpha N}}{\partial X_{\beta P}} \frac{\partial X_\beta^M}{\partial X_\alpha^P} \right) . \quad (3.13)$$

An extensive investigation of the transformations induced on the fields after solving the strong section condition⁵ has been made in [17, 25]. Interpreting these transformations as patching conditions, one can show that if one considers either

$$x_\alpha^i = x_{\alpha\beta}^i(x_\beta^j), \quad y_{\alpha i} = y_{\beta i} \quad (3.14)$$

or shift transformations

$$x_\alpha^i = x_\beta^i, \quad y_{\alpha i} = y_{\beta i} - \zeta_{\alpha\beta i} \quad (3.15)$$

of the double coordinates, then the 2-form gauge potential b of the NS-NS 3-form field strength transforms either as a 2-form, $b_\alpha = b_\beta$ or as $b_\alpha = b_\beta + d\zeta_{\alpha\beta}$, where in the latter case the x transformation is the identity

Clearly, none of these two patching conditions are satisfactory. If b transforms as a 2-form, then one has to restrict H to be exact. The shift transformation is also not sufficient as we know that string theory admits solutions which require that M patches with non-trivial transition functions.

⁴For some mathematical aspects of DFT see eg [29] and also [30] for explicit backgrounds.

⁵If the strong section condition is relaxed, then the generalized transformations on the forms do not commute with the exterior derivative. So closed forms do not transform to closed forms.

The authors of [17] have also investigated the combined transformation which includes diffeomorphisms of the spacetime M combined with shift transformations of the y coordinates which as a patching condition reads

$$x_\alpha^i = x_{\alpha\beta}^i(x_\beta^j) , \quad y_{\alpha i} = y_{\beta i} - \zeta_{\alpha\beta i} . \quad (3.16)$$

In turn these give rise to the patching condition

$$\begin{aligned} (b_\alpha)_{ij} &= \frac{\partial x_\beta^k}{\partial x_\alpha^i} \frac{\partial x_\beta^l}{\partial x_\alpha^j} \left((b_\beta)_{kl} + \frac{1}{2} \left(\frac{\partial}{\partial x_\beta^k} \zeta_{\alpha\beta, l} - \frac{\partial}{\partial x_\beta^l} \zeta_{\alpha\beta, k} \right) \right) \\ &+ \frac{1}{2} \left(\frac{\partial x_\beta^k}{\partial x_\alpha^i} \frac{\partial}{\partial x_\beta^k} \zeta_{\alpha\beta, j} - \frac{\partial x_\beta^k}{\partial x_\alpha^j} \frac{\partial}{\partial x_\beta^k} \zeta_{\alpha\beta, i} \right) , \end{aligned} \quad (3.17)$$

for the 2-form gauge potential b .

It is clear from (3.16) that consistency on triple overlaps on the spacetime requires that

$$\zeta_{\alpha\beta} + \zeta_{\beta\gamma} + \zeta_{\gamma\alpha} = 0 . \quad (3.18)$$

If this is the case, a consequence of the lemma proven in section 3.2 implies that there are 1-forms $\{u_\alpha\}$ defined on the open sets $\{U_\alpha\}$ such that

$$\zeta_{\alpha\beta} = -u_\alpha + u_\beta , \quad (3.19)$$

where

$$u_\alpha = \sum \rho_\gamma \zeta_{\gamma\alpha} . \quad (3.20)$$

Thus (3.17) can be rewritten as

$$\begin{aligned} ((b_\alpha)_{ij} + \frac{1}{2} (du_\alpha)_{ij}) dx_\alpha^i \wedge dx_\alpha^j &= ((b_\beta)_{ij} + \frac{1}{2} (du_\beta)_{ij}) dx_\beta^i \wedge dx_\beta^j \\ &- \frac{1}{2} \frac{\partial}{\partial x_\beta^i} (u_{\alpha k} \frac{\partial x_\alpha^k}{\partial x_\beta^j}) dx_\beta^i \wedge dx_\beta^j + \frac{1}{2} \frac{\partial}{\partial x_\beta^i} (u_{\beta k} \frac{\partial x_\alpha^k}{\partial x_\beta^j}) dx_\beta^i \wedge dx_\beta^j , \end{aligned} \quad (3.21)$$

or equivalently,

$$b_\alpha - b_\beta = -\frac{1}{2} du_\alpha + \frac{1}{2} du_\beta - \frac{1}{2} d\tilde{u}_\alpha + \frac{1}{2} d\tilde{u}_\beta . \quad (3.22)$$

As a result

$$\tilde{b}_\alpha \equiv b_\alpha + \frac{1}{2} du_\alpha + \frac{1}{2} d\tilde{u}_\alpha = b_\beta + \frac{1}{2} du_\beta + \frac{1}{2} d\tilde{u}_\beta \equiv \tilde{b}_\beta . \quad (3.23)$$

So up to a gauge transformation, b can be made globally defined on M and $H = db = d\tilde{b}$. Thus H is exact.

One therefore concludes that the patching conditions induced on the spacetime from the generalized coordinate transformations of DFT after solving the strong section condition imply that H is an exact form.

This conclusion cannot be satisfactory as we know string theory has as solutions space-times that admit non-trivial patching and a closed but not exact form H . The restriction that H is exact is a direct consequence of the introduction of the new coordinates y , their transition functions and their relation to the patching conditions of the 2-form gauge potentials as this is implied from the generalized coordinate transformations.

It is worth pointing out that H is not restricted to be exact in the context of generalized geometry⁶. In generalized geometry, there are no additional coordinates that have to be patched. The analogous consistency condition which arises on triple overlaps reads

$$d(a_{\alpha\beta}^1 + a_{\beta\gamma}^1 + a_{\gamma\alpha}^1) = 0, \quad (3.24)$$

and it does not impose additional restrictions on the transition functions of H . The above condition is always satisfied as it can be seen from the analysis of the section 3.1.

3.4 Seeking a consistent patching

It is not apparent how to reconcile the transition functions of the double space with those of the spacetime, and the patching conditions of the 3-form field strength H without imposing additional conditions on the fluxes. Any choice of transition functions for the double space of the type

$$x_\alpha^i = x_{\alpha\beta}^i(x_\beta^j), \quad y_{\alpha i}^1 = y_{\beta i}^1 - \zeta_{\alpha\beta i}^1 \quad (3.25)$$

where the patching conditions $a_{\alpha\beta}^1$ of H are linear combination of $\zeta_{\alpha\beta}^1$'s and the combinatorial rules denoted are as those of tensor calculus, will lead to the conclusion that H is exact. Another indication that such a direct approach may not be fruitful is that there is nowhere use of the quantisation condition $[H] \in H^3(M, \mathbb{Z})$ which has a central role in the exploration of the $U(1)$ paradigm.

In appendix A, a modification of the patching conditions of the y coordinates was proposed and it was shown that there are no additional restrictions at triple overlaps. If this modification is considered, the double space patches consistently and it has some attractive features like its tangent bundle is an extension of the tangent bundle of the spacetime. However, the topological geometrisation of H fails, ie when H is pulled back on the double space is not exact which is in conflict with the $U(1)$ field paradigm. Perhaps this is not surprising as to be able to topologically geometrize H , the additional coordinates have to exhibit non-trivial topology related to the class of H in $H^3(M, \mathbb{Z})$. In the conclusions, we propose a general framework where all these questions may be addressed.

4 Revisiting examples

4.1 Reviewing the double space of 3-torus with constant H flux

In this section, we review the construction of the double space of T^3 with constant H flux. This example is the most relevant one of the three examples presented in [17] to explore

⁶Generalized geometry does not require that H represents a class in $H^3(M, \mathbb{Z})$ and so it does not capture the analogue of the Dirac quantisation condition in string theory and M-theory. It also does not obey the topological geometrisation condition unlike the $U(1)$ field paradigm.

the patching of the double spaces. This is because the consistency of the construction can be checked at the level of the transition functions for the coordinates of the double space. In the other two examples instead, the focus is on the consistency of the patching of generalized tensors.

After some relabeling of the coordinates and in geometric units, the model can be described as follows. We equip T^3 with three angular “coordinates”⁷ $(\theta^1, \theta^2, \theta^3)$ such that $0 \leq \theta^1, \theta^2, \theta^3 < 2\pi$ and take as an H flux $H = N d\theta^1 \wedge d\theta^2 \wedge d\theta^3$. The quantization of flux

$$\frac{1}{(2\pi)^2} \int H \in \mathbb{Z} \quad (4.1)$$

requires that $2\pi N \in \mathbb{Z}$.

The construction of double space in [17] proceeds as follows. First the Poincaré lemma is used to locally solve for the gauge potential b of H as

$$b = N \theta^3 d\theta^1 \wedge d\theta^2 . \quad (4.2)$$

Then it is noticed that

$$b(2\pi) - b(0) = 2\pi N d\theta^1 \wedge d\theta^2 . \quad (4.3)$$

This apparent lack of periodicity for b is compensated by a gauge transformation $b' = b - d\xi$, where

$$\xi = 2\pi N \theta^1 d\theta^2 , \quad (4.4)$$

and so

$$b'(2\pi) = b(2\pi) - 2\pi N d\theta^1 \wedge d\theta^2 = b(0) . \quad (4.5)$$

This gauge transformation is then used to define the coordinate transformations of the DFT background as

$$\begin{aligned} \theta'^1 &= \theta^1 , & \theta'^2 &= \theta^2 , & \theta'^3 &= \theta^3 , \\ \psi'^1 &= \psi^1 , & \psi'^2 &= \psi^2 - 2\pi N \theta^1 , & \psi'^3 &= \psi^3 , \end{aligned} \quad (4.6)$$

according to (3.15), where (ψ^1, ψ^2, ψ^3) are the dual coordinates which are taken as the coordinates of a dual torus \tilde{T}^3 .

Below we shall demonstrate that the construction of the double space relies for consistency on the existence of a particular atlas on T^3 . As a result, it has limited applicability. In the process of proving this, several silent features of the above construction will become apparent.

⁷Note these are not coordinates in the sense of manifold theory. Instead they should be thought as labels that denote the points on the circle.

4.2 Patching the double space of 3-torus with constant H

4.2.1 Patching conditions

To describe the construction of double spaces from a patching point of view, we shall first describe an atlas on the circle S^1 which in turn will induce an atlas on the 3-torus T^3 . For the former, we cover S^1 with two patches $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ such that

$$\begin{aligned}\varphi_1 : U_1 \subset S^1 &\rightarrow \left(-\frac{\pi}{4}, \frac{5\pi}{4}\right), & \varphi_1((s, t)) &= x_1, \\ \varphi_2 : U_2 \subset S^1 &\rightarrow \left(-\frac{5\pi}{4}, \frac{\pi}{4}\right), & \varphi_2((s, t)) &= x_2,\end{aligned}\tag{4.7}$$

where U_1 is the open set which includes the north pole and U_2 is the open set which includes the south pole, and we have solved the algebraic equation $s^2 + t^2 = 1$ of S^1 as $s = \cos x_1, t = \sin x_1$ and $s = \cos x_2, t = \sin x_2$.

The intersection $\varphi_1(U_1 \cap U_2) = (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$ or equivalently $\varphi_2(U_1 \cap U_2) = (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (-\frac{5\pi}{4}, -\frac{3\pi}{4})$. Therefore the transition functions are $x_2 = \varphi_2 \circ \varphi_1^{-1}(x_1)$ which give

$$\begin{aligned}x_2 &= x_1, & \text{on } \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ x_2 &= x_1 - 2\pi, & \text{on } \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right).\end{aligned}\tag{4.8}$$

It is convenient to write the above transition function as

$$x_2 = x_1 + 2n_x\pi,\tag{4.9}$$

with the understanding that $n_x = 0$ on $(-\frac{\pi}{4}, \frac{\pi}{4})$ and $n_x = -1$ on $(\frac{3\pi}{4}, \frac{5\pi}{4})$. Of course the transition function $\varphi_1 \circ \varphi_2^{-1}$ is the inverse of $\varphi_2 \circ \varphi_1^{-1}$, ie $x_1 = x_2 - 2n_x\pi$. Observe that

$$dx_1 = dx_2,\tag{4.10}$$

and so dx is a globally defined closed but not exact 1-form on S^1 as expected. We refer to x as the angular manifold coordinates of S^1 to distinguish them from the θ s of the previous section.

Using the above atlas on S^1 , one can induce a manifold structure on T^3 as follows. First, we introduce two patches as above for each one of the three circles, ie $\{(U_i, \varphi_i)|i = 1, 2\}$, $\{(V_j, \lambda_j)|j = 1, 2\}$ and $\{(W_k, \mu_k)|k = 1, 2\}$, and then take their products. The atlas on T^3 is $\{(U_i \times V_j \times W_k, \varphi_i \times \lambda_j \times \mu_k)|i, j, k = 1, 2\}$, ie T^3 is covered by 8 patches. It is convenient to define $U_{ijk} = U_i \times V_j \times W_k$ and $\varphi_{ijk} = \varphi_i \times \lambda_j \times \mu_k$.

In this atlas on T^3 , the H flux can now be written as $H = Ndx \wedge dy \wedge dz$, where x, y and z are the angular manifold coordinates for the three circles. To continue we use the Poincaré lemma for each of the open sets U_{ijk} to identify the gauge potential of H as

$$b_{ijk} = \frac{N}{3}(x_i dy \wedge dz - y_j dx \wedge dz + z_k dx \wedge dy).\tag{4.11}$$

Note that the patch Greek labels α of section 3.1 have been replaced with the multi-labels ijk . On the double overlaps $U_{i_1 j_1 k_1} \cap U_{i_2 j_2 k_2}$, we have that

$$b_{i_1 j_1 k_1} = b_{i_2 j_2 k_2} + \frac{N}{3}2\pi[n_x(i_1 - i_2)dy \wedge dz$$

$$-n_y(j_1 - j_2)dx \wedge dz + n_z(k_1 - k_2)dx \wedge dy] . \quad (4.12)$$

As a result, the 1-form transition functions $a_{i_1 j_1 k_1, i_2 j_2 k_2}^1$ of section 3.1 can be chosen as

$$\begin{aligned} a_{i_1 j_1 k_1, i_2 j_2 k_2}^1 = & \frac{N}{12} 2\pi \{ [n_x(i_1 - i_2)(y_{j_1} + y_{j_2}) - n_y(j_1 - j_2)(x_{i_1} + x_{i_2})]dz \\ & + [-n_x(i_1 - i_2)(z_{k_1} + z_{k_2}) + n_z(k_1 - k_2)(x_{i_1} + x_{i_2})]dy \\ & + [n_y(j_1 - j_2)(z_{k_1} + z_{k_2}) - n_z(k_1 - k_2)(y_{j_1} + y_{j_2})]dx \} . \end{aligned} \quad (4.13)$$

This expression respects the symmetries of the transition functions that arise from the exchange of the patch labels.

To continue, we compute da^0 at triple overlaps to find that

$$\begin{aligned} da_{i_1 j_1 k_1, i_2 j_2 k_2, i_3 j_3 k_3}^0 = & \frac{N}{12} (2\pi)^2 \{ n_x n_y [i_1(j_2 - j_3) - j_1(i_2 - i_3) - i_2(j_1 - j_3) \\ & + j_2(i_1 - i_3) + i_3(j_1 - j_2) - j_3(i_1 - i_2)]dz \\ & + n_x n_z [-i_1(k_2 - k_3) + i_2(k_1 - k_3) + k_1(i_2 - i_3) \\ & - k_2(i_1 - i_3) - i_3(k_1 - k_2) + k_3(i_1 - i_2)]dy \\ & + n_y n_z [j_1(k_2 - k_3) - j_2(k_1 - k_3) - k_1(j_2 - j_3) \\ & + k_2(j_1 - j_3) - k_3(j_1 - j_2) + j_3(k_1 - k_2)]dx \} , \end{aligned} \quad (4.14)$$

and n at quadruple intersections to find that

$$\begin{aligned} n_{i_1 j_1 k_1, i_2 j_2 k_2, i_3 j_3 k_3, i_4 j_4 k_4} = & \frac{N}{24} (2\pi)^3 n_x n_y n_z \{ [(i_1 j_2 - j_1 i_2)(k_3 - k_4) + (j_1 i_3 - j_3 i_1)(k_2 - k_4) \\ & + (i_2 j_3 - j_2 i_3)(k_1 - k_4) + (j_2 i_4 - i_2 j_4)(k_1 - k_3) + (j_4 i_1 - j_1 i_4)(k_2 - k_3) + (i_3 j_4 - j_3 i_4)(k_1 - k_2)] \\ & - [(i_1 k_2 - k_1 i_2)(j_3 - j_4) + (k_1 i_3 - k_3 i_1)(j_2 - j_4) + (i_2 k_3 - k_2 i_3)(j_1 - j_4) + (k_2 i_4 - i_2 k_4)(j_1 - j_3) \\ & + (k_4 i_1 - k_1 i_4)(j_2 - j_3) + (i_3 k_4 - k_3 i_4)(j_1 - j_2)] + [(j_1 k_2 - k_1 j_2)(i_3 - i_4) + (k_1 j_3 - k_3 j_1)(i_2 - i_4) \\ & + (j_2 k_3 - k_2 j_3)(i_1 - i_4) + (k_2 j_4 - k_4 j_2)(i_1 - i_3) + (k_4 j_1 - k_1 j_4)(i_2 - i_3) \\ & + (j_3 k_4 - k_3 j_4)(i_1 - i_2)] \} . \end{aligned} \quad (4.15)$$

Note that da^0 has constant components. This is an atlas dependent statement. da^0 will not have constant components in another atlas. Of course n is constant as expected independent of the choice of atlas.

Denoting the double coordinates with X, Y, Z , one can according to (3.15) impose the patching condition

$$\begin{aligned} X_{i_1 j_1 k_1} dx + Y_{i_1 j_1 k_1} dy + Z_{i_1 j_1 k_1} dz = & X_{i_2 j_2 k_2} dx + Y_{i_2 j_2 k_2} dy \\ & + Z_{i_2 j_2 k_2} dz - a_{i_1 j_1 k_1, i_2 j_2 k_2}^1 . \end{aligned} \quad (4.16)$$

Without further assumptions on X, Y, Z , a check of the consistency of the above patching condition at triple overlaps will lead to the requirement that

$$da^0 = 0 . \quad (4.17)$$

This is clearly a contradiction as $da^0 \neq 0$ which can be seen from (4.14). This is in agreement with the proof presented in sections 3.1, 3.2 and 3.3.

However for this particular example, one can also impose as patching condition

$$\begin{aligned} & \left(X_{i_1 j_1 k_1} dx + Y_{i_1 j_1 k_1} dy + Z_{i_1 j_1 k_1} dz - X_{i_2 j_2 k_2} dx - Y_{i_2 j_2 k_2} dy \right. \\ & \quad \left. - Z_{i_2 j_2 k_2} dz + a_{i_1 j_1 k_1, i_2 j_2 k_2}^1 \right) = 0 \mod \mathbb{Z}(\ell, \ell, \ell) , \end{aligned} \quad (4.18)$$

instead of that in (4.16). Choosing $\ell = \frac{N}{3}(2\pi)^2$, the inconsistency at the triple overlaps is removed and the patching becomes consistent.

Note however that this modification of the transition functions at triple overlaps goes beyond the suggestion of [26] that the transition functions are just the local diffeomorphisms of the double space. To put it in another way, the periodic identification of the dual coordinates does not follow from the patching conditions of the double space alone as constructed from the patching conditions of T^3 and the 2-form gauge potential. Instead it has to be imposed by hand.

It remains to investigate whether a modification of the patching condition as in (4.18) is always possible. This will be explored below.

4.2.2 Consistency of the patching

The modification of the patching condition as in (4.18) comes at a cost. First, the construction is atlas dependent. If one uses another atlas on T^3 , the components of da^0 are functions of the coordinates instead of constants. As a result, the modification of the patching condition as in (4.18) is not valid.

Nevertheless, one could argue that since there is an atlas on T^3 as in (4.7) and (4.8) that the patching condition (4.18) is valid, one can perform a diffeomorphism on T^3 which can take T^3 with respect to any atlas to T^3 with the (4.7) and (4.8) atlas. Such a diffeomorphism $x' = x'(x)$ will transform the dual coordinates schematically as

$$X' = \frac{\partial x}{\partial x'} X , \quad Y' = \frac{\partial x}{\partial x'} Y , \quad Z' = \frac{\partial x}{\partial x'} Z . \quad (4.19)$$

However, such a transformation is not allowed within the DFT as does not solve the strong section condition.

Furthermore it is worth contrasting the patching conditions (4.18) with those of (2.4) in section 2 for the 2-form paradigm. The modification of the combinatorial law in (2.4) and the subsequent consistency of patching at triple overlaps are atlas independent. As a result, the mechanism in section 2 can apply to any background and any closed 2-form provided that represents an integral class. This is not the case for the patching in (4.18) as we have seen.

The above comparison also explains the difficulty in constructing the double spaces of generic string backgrounds. If one insists of using a modification of the patching condition as in (4.18), then one has to prove that the string background admits an atlas such that at all triple overlaps da^0 has constant components. It is not apparent that such an atlas exists for general manifolds with a closed 3-form. So the modification of the patching conditions (4.18) may be limited to spaces with toroidal topology.

To summarize, the construction of the double space for T^3 with H flux background depends on the particular atlas we have chosen on T^3 . Therefore, the whole construction

is rather special attached to the details of a manifold structure on T^3 . Since it depends on the choice of the atlas, the construction is not general covariant and as a result (4.18) does not generalize to all spaces in a way similar to (2.4).

5 Patching k-forms and exceptional coordinates

5.1 Transition functions of closed k-forms

The patching conditions of any closed k-form ω^k , $k > 3$, on a manifold M can be found in a similar way as those for closed 3-forms in section 3.1. For this, it is convenient to use the difference operator δ of the Čech-de Rham theory defined as

$$\delta\chi_{\alpha_0\alpha_1\dots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \chi_{\alpha_0\alpha_1\dots\hat{\alpha}_i\dots\alpha_{p+1}} , \quad (5.1)$$

where χ is a q-form defined at $p+1$ -overlaps and the caret denotes omission. $\delta\chi$ is a q-form defined at $(p+2)$ -overlaps and it is understood that in the right-hand-side of the above equation χ is restricted on $U_{\alpha_0} \cap \dots \cap U_{\alpha_{p+1}}$. Observe that $\delta^2 = 0$ and $d\delta = \delta d$. For more details on the properties of δ see eg [34].

Given now a globally defined closed k-form on M , $k \geq 3$, we use the Poincaré lemma to write $\omega_\alpha^k = dC_\alpha^{k-1}$. Then we obtain the patching conditions⁸

$$\delta C_{\alpha_0\alpha_1}^{k-1} = da_{\alpha_0\alpha_1}^{k-2} , \quad \delta a_{\alpha_0\alpha_1\alpha_2}^{k-2} = da_{\alpha_0\alpha_1\alpha_2}^{k-3} , \quad (5.2)$$

and so on till

$$\delta a_{\alpha_0\dots\alpha_{k-1}}^1 = da_{\alpha_0\dots\alpha_{k-1}}^0 , \quad \delta a_{\alpha_0\dots\alpha_k}^0 = n_{\alpha_0\dots\alpha_k} , \quad (5.3)$$

where $n_{\alpha_0\dots\alpha_k}$ are constants. If $n_{\alpha_0\dots\alpha_k} \in 2\pi\mathbb{Z}$, then ω^k represents a class in $H^k(M, \mathbb{Z})$.

The $(k-1)$ -form potentials $\{C_\alpha^{k-1}\}$ are not uniquely defined. In particular, there are defined up to a gauge transformation as

$$C_\alpha^{k-1} \rightarrow C_\alpha^{k-1} + d\chi_\alpha^{k-2} . \quad (5.4)$$

Similarly, the remaining patching conditions are defined up to gauge transformations of the type

$$a_{\alpha_0\dots\alpha_q}^p \rightarrow a_{\alpha_0\dots\alpha_q}^p + d\chi_{\alpha_0\dots\alpha_q}^{p-1} + (\delta\psi^p)_{\alpha_0\dots\alpha_q} \quad (5.5)$$

for some $\chi_{\alpha_0\dots\alpha_q}^{p-1}$ and $\psi_{\alpha_0\dots\alpha_{q-1}}^p$, where $p+q = k-1$. This ambiguity in the definition of patching conditions of ω^k is the only one allowed consistent with $d\omega^k = 0$ and the transition functions of M .

⁸ There are some sign differences in the definitions of a 's as compared with those given in section 3.

5.2 Exact k-forms and patching conditions

As in the 3-form case utilizing the ambiguity in the definition of the patching conditions (5.5), it is possible to show that if a^p , $p = 0, \dots, k-2$ satisfies the cocycle condition

$$(\delta a^p)_{\alpha_0 \dots \alpha_{q+1}} = 0 , \quad (5.6)$$

then ω^k is exact.

In what follows, it suffices to prove this for the $a_{\alpha_0 \alpha_1}^{k-2}$ as those are responsible for the patching a k-form at double overlaps, and potentially can be used to construct the exceptional generalized spaces. The proof is similar to that we have given for 3-forms. In particular, suppose that

$$(\delta a^{k-2})_{\alpha_0 \alpha_1 \alpha_2} = 0 . \quad (5.7)$$

Then define

$$\tilde{C}_{\alpha_0}^{k-1} = C_{\alpha_0}^{k-1} - d \left(\sum_{\gamma} \rho_{\gamma} a_{\gamma \alpha_0}^{k-2} \right) . \quad (5.8)$$

Clearly $d\tilde{C}_{\alpha_0}^{k-1} = dC_{\alpha_0}^{k-1} = \omega_{\alpha_0}^k$ as \tilde{C}^{k-1} and C^{k-1} are related up to a gauge transformation.

Moreover \tilde{C}^{k-1} is a globally defined (k-1)-form as

$$\begin{aligned} \delta(\tilde{C}^{k-1})_{\alpha_0 \alpha_1} &= (\delta C^{k-1})_{\alpha_0 \alpha_1} - d \left(\sum_{\gamma} \rho_{\gamma} (a_{\gamma \alpha_0}^{k-2} - a_{\gamma \alpha_1}^{k-2}) \right) \\ &= d a_{\alpha_0 \alpha_1}^{k-2} - d \sum_{\gamma} (\rho_{\gamma} a_{\alpha_0 \alpha_1}^{k-2}) = 0 , \end{aligned} \quad (5.9)$$

where we have used (5.7) and that $\sum_{\gamma} \rho_{\gamma} = 1$.

5.3 Seeking a consistent patching for EFT

For EFT there is not an analogue of the patching conditions of [25] available for DFT. Instead the constructions have been based on using infinitesimal symmetries generated by generalized Lie derivatives, see eg [10, 11, 13, 14, 26–28] for detailed descriptions. One expects that whatever the final form of the finite transformations are for EFT, these will generate both the transition functions of the underlying spacetime and the patching conditions of the form field strengths of the theory. After solving the strong section condition, the two must be related. Following the analogous analysis for DFT, one may hypothesize that the transition functions of the exceptional space read as

$$x_{\alpha}^i = x_{\alpha\beta}(x_{\beta}^j) , \quad y_{\alpha_0}^{k-2} - y_{\alpha_1}^{k-2} = -\zeta_{\alpha_0 \alpha_1}^{k-2} , \quad (5.10)$$

where x and y^{k-2} are the spacetime and additional coordinates, respectively, and the patching data $a_{\alpha\beta}^{k-2}$ of the k-form field strength are a linear combination of $\zeta_{\alpha_0 \alpha_1}^{k-2}$. If this is the case, then again consistency at triple overlaps will require that ω^k is exact.

In appendix A, we propose a modification of the patching conditions (5.10) which resolves the restriction at triple overlaps. However, it does not topologically geometrize

the k -form field strength. Similar constructions can be made in theories that we include the dual fields as demonstrated in appendix A for 11-dimensional supergravity. In section 6, we make an alternative proposal how a priori topologically geometrize k -forms field strengths based on K-theory and homotopy theory.

The above result does not hold for exceptional generalized geometries, ie those that no new coordinates are introduced in addition to those of spacetime. This is because they do not require the condition (5.7) but instead

$$d\delta(a^{k-2})_{\alpha_0\alpha_1\alpha_2} = 0 . \quad (5.11)$$

This does not introduce a restriction on the patching conditions of form field strengths.

6 Summary and outlook

We have shown that the patching conditions of DFT as arise from generalize coordinate transformations after solving the strong section condition imply that the NS-NS 3-form field is an exact 3-form. A similar conclusion may hold in the context of EFT under some plausible assumptions regarding the relation between the transition functions of additional exceptional coordinates and the patching data of the form field strengths. We have also explored some alternative possibilities. These resolve some of the difficulties, like the restriction on the form field strengths to be exact, but they do not obey the topological geometrisation condition which is one of the key properties that the $U(1)$ field paradigm.

Furthermore we revisited the double space construction of T^3 with H -flux background from the patching conditions point of view. If the combinatorial law of the transition functions is not altered at triple overlaps, the construction is inconsistent. However, there is a modification of the combinatorial law at triple overlaps which allows for a consistent construction of the double space. But this modification depends on the choice of an atlas on T^3 , ie it is not general covariant, and so as a result cannot be generalized to generic backgrounds with H -fluxes.

To introduce new coordinates that extend the spacetime in a consistent way without any further conditions on the fields, like exactness of the form field strengths, one may try to generalize some aspects of the 2-form paradigm reviewed in section 2. Some of the directions that can be pursued are the following.

- To modify the combinatorial law of transition functions.
- To modify the transition functions.
- To introduce topology on the generalized spaces.

It is clear that the 2-form paradigm is consistent because it has been possible to appropriately modify the combinatorial law of the transition functions. Of course, this has been achieved under the additional requirement that ω^2 represents a class in $H^2(M, \mathbb{Z})$. Although this imposes a restriction on the transition functions of ω^2 , this restriction is required by the Dirac quantisation condition. In the context of DFTs and EFTs, this is

an indication that non-commutative geometry has a role as it is not possible to alter the combinatorial law of transition functions in a straight forward way, see also [33].

As we have seen a mild modification of the patching conditions of the form field strengths, eg as linear functions of the transition functions of the additional coordinates, leads to the conclusion that consistency at triple overlaps requires that the form field strengths are exact. However as we explain in appendix A, there is a modification of the transition functions of the additional coordinates such that the double and exceptional spaces are consistent at triple overlaps without any restrictions on the patching conditions of the form field strengths. However, as such a modification has its problems, like for example the topological geometrisation condition does not hold. Moreover there is no use of the cohomological analogue the Dirac quantisation condition $[\omega^k] \in H^k(M, \mathbb{Z})$ in the construction of the double of exceptional space above which has a central role in the $U(1)$ field paradigm.

It is clear from the above that whatever the construction of these extended spaces is the additional coordinates have to have a non-trivial topology. This is the only way that both the topological geometrisation and the cohomological analogue of the Dirac quantisation $[\omega^k] \in H^k(M, \mathbb{Z})$ conditions can be utilized to construct these spaces. It is not a priori apparent how this can be done or whether a consistent construction is possible for all cases of interest, beyond those of toroidal compactifications, but a way to proceed is as follows.

One of the difficulties in adapting the $U(1)$ field paradigm in the context of string theory and M-theory is that after applying a duality transformation the spacetime may change as a manifold at the same time as the fields of the theory. One way to incorporate this into the construction of extended spaces is as follows. Suppose that $(M_I, \mathcal{F}_I)_{I \in \mathcal{I}}$ be a family of spacetimes M_I with field content \mathcal{F}_I such that any two pairs (M_I, \mathcal{F}_I) and (M_J, \mathcal{F}_J) are related by a duality transformation D_{IJ} , $D_{IJ} : (M_I, \mathcal{F}_I) \rightarrow (M_J, \mathcal{F}_J)$. One way to geometrize the data (M_I, \mathcal{F}_I) and D_{IJ} is to assume that there is a space CM , a C-fold, and maps $\pi_I : CM \rightarrow M_I$ such that

1. $(\pi_I)_* TCM = TM_I$,
2. the duality transformation D_{IJ} has a lift \tilde{D}_{IJ} on CM which is implemented with a transformation which respects the topological and geometric properties of the C-fold, CM , and $D_{IJ} \circ \pi_I = \pi_J \circ \tilde{D}_{IJ}$, and
3. as a minimal requirement assert that the pull back $\pi_I^* \omega_I$ of all the fields $\omega_I \in \mathcal{F}_I$ on M_I which represent non-trivial classes $[\omega_I]$ in cohomology $H^*(M_I)$ represent the trivial class $[\pi_I^* \omega_I] = 0$ in $H^*(CM)$.

The last property is the implementation of the topological geometrisation condition which should appear as a weak restriction for carrying out the geometrisation programme. It would be necessary to impose additional conditions on CM but the above three conditions can serve as a minimal requirement.

It is not apparent whether the construction of C-folds CM would be possible for all backgrounds in string theory and M-theory. However, there are constructions which satisfy some of the three conditions mentioned above. For this consider the trivial case, where

the duality transformation D_{IJ} is the identity and let us view CM as a fibration over M . There are many ways that the construction of a CM can be achieved which satisfies the topological geometrisation property. One way is K-theory which generalizes the 2-form paradigm. Suppose that one tries to construct a C-fold, CM_{ω^4} , of a spacetime that has field content a closed 4-form ω^4 . If ω^4 is represented by the second Chern class or the first Pontryagin class of a complex or real vector bundle E , respectively, then as a CM can be taken as the principal bundle P that E is associated to. In such a case ω^4 will represent an integral class and the pull-back of ω^4 on P will represent the trivial class. It may be possible to find amongst the K-theory class which resolves the topological geometrisation condition a representative which will also exhibit the local geometric requirements as expected from DFTs and EFTs. K-theory has appeared before in string theory and M-theory [35], see also [36] and references within. Therefore, it may not be a surprise that it could be also applicable in this context. Moreover the tangent space of all fibrations are extensions of that of the base space, and so $T(CM_{\omega^4}) \rightarrow \pi^*TM \rightarrow 0$. This is one of the properties expected for generalized manifolds.

It is tantalizing for example that one can construct C-folds for the $AdS_7 \times S^4$ near horizon geometry of M5-branes. It is known that for every class in $[\omega] \in H^4(S^4, \mathbb{Z})$ there is a complex vector bundle E such that $c_2(E) = [\omega]$. This follows from the isomorphism of $\tilde{K}(S^4)$ with $H^4(S^4, \mathbb{Z})$, see eg [37]. The associated real bundle is an $SO(4)$ bundle with Pontryagin class $p_1(E_{\mathbb{R}}) = 2c_2(E)$. In fact all the $SO(4)$ bundles are classified by two integers n, m as $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ and $p_1 = 2(n - m)$ [38]. Considering the associated $SO(4)$ principal bundles, observe that $T_e SO(4) = \mathfrak{so}(4) = \Lambda^2(\mathbb{R}^4)$. So the tangent space of the fibres provide a local model for the 2-forms on S^4 . This is in line with the expectation that the additional coordinates of this example are locally modeled by 2-forms on S^4 .

Furthermore, there are homotopy constructions which can implement the topological geometrisation property. One such construction is the Whitehead towers, see eg [34]. These are given by successive fibrations

$$K(\pi_n, n-1) \rightarrow Y_{n+1} \rightarrow Y_n, \quad (6.1)$$

where $\pi_n = \pi_n(X)$, $Y_1 = X$, $\pi_k(Y_n) = 0$ for $k < n$ and $\pi_k(Y_n) = \pi_k(X)$ for $k > n$, and $K(\pi_n, n-1)$ are Eilenberg-MacLane spaces. Therefore, Y_2 is the universal cover of X , $\pi_1(Y_3) = \pi_2(Y_3) = 0$ and so on. Thus if $[\omega]$ represents a class in the first non-vanishing cohomology group of X , say $H^n(X, \mathbb{Z})$, this class becomes trivial in Y_{n+1} implementing the topological geometrisation condition.

Although the homotopy constructions above based on the Whitehead towers can implement the topological geometrisation condition, the spaces involved are infinite dimensional. It is not apparent that one can generically find a finite dimensional model which has all the required properties as for the $AdS_7 \times S^4$ example presented above. Nevertheless the above homotopy constructions use path spaces and so may admit a string theoretic interpretation.

This procedure based on Whitehead towers can be applied to implement the topological geometrisation condition on a Calabi-Yau mirror pair. Suppose X_1 is a Calabi-Yau manifold and X_2 its mirror. Consider $X_1 \times X_2$. The first non-vanishing cohomology group is $H^2(X_1 \times X_2) = H^2(X_1, \mathbb{Z}) \oplus H^2(X_2, \mathbb{Z})$. To implement the topological geometrisation condition, it

suffices to consider Y_4 . By construction $H^2(Y_4, \mathbb{Z}) = H^3(Y_4, \mathbb{Z}) = 0$ and Y_4 fibres above $X_1 \times X_2$ and so above both X_1 and X_2 . Therefore the H^2 and H^3 cohomologies classes of both X_1 and X_2 become trivial in Y_4 . It will be of interest to determine the action of \tilde{D}_{12} on Y_4 , where D_{12} is the mirror symmetry of X_1 and X_2 .

Therefore there are some suggestions how to construct generalized manifolds which allow for form field strengths to represent non-trivial cohomology classes. However, it is less apparent which will be the most fruitful way to proceed. The expectation is that the full theory at the end will combine many of the local computations that have been done so far with the global aspects that many of the backgrounds have in a consistent way.

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A Modifying transition functions

A.1 New transition functions and k-forms

Let ω^k be closed k-form on M , $k \geq 3$, with transition functions as those defined in section 4. To define \tilde{M}_{ω^k} , we introduce coordinates (x, y^{k-2}) , where x are the coordinates of M , and impose the transition functions

$$x_{\alpha_0} = f_{\alpha_0 \alpha_1}(x_{\alpha_1}) , \quad \delta y_{\alpha_0 \alpha_1}^{k-2} \equiv -y_{\alpha_0}^{k-2} + y_{\alpha_1}^{k-2} = a_{\alpha_0 \alpha_1}^{k-2} - \sum_{\beta} \rho_{\beta} da_{\beta \alpha_0 \alpha_1}^{k-3} . \quad (\text{A.1})$$

Since $\delta^2 = 0$, consistency at triple overlaps requires that

$$\delta(a^{k-2} - \sum_{\beta} \rho_{\beta} da_{\beta}^{k-3})_{\alpha_0 \alpha_1 \alpha_2} = 0 \quad (\text{A.2})$$

which is satisfied as it can be shown after a calculation using (5.2).

It is clear that there is a projection π from \tilde{M}_{ω^k} onto M and so it can be thought that \tilde{M}_{ω^k} is a bundle over M . However as in the case of 3-forms there is no natural global section of M in \tilde{M}_{ω^k} , eg the zero section is not preserved by the transition functions.

Since \tilde{M}_{ω^k} is a manifold, one can investigate the tangent as well as all the other tensor bundles in the standard way. We shall do this for \tilde{M}_{ω^3} as the generalization to \tilde{M}_{ω^k} is straightforward. To do this let us write the patching conditions (5.10) explicitly as

$$x_{\alpha}^i = f_{\alpha\beta}^i(x_{\beta}^j) , \quad (y_{\alpha}^1)_i - (y_{\beta}^1)_k \frac{\partial x_{\beta}^k}{\partial x_{\alpha}^i} = (\tilde{a}_{\alpha\beta}^1)_i \quad (\text{A.3})$$

where

$$\tilde{a}_{\alpha\beta}^1 = -a_{\alpha\beta}^1 + \sum_{\delta} \rho_{\delta} da_{\delta\alpha\beta}^0 . \quad (\text{A.4})$$

Note that the second patching condition in (A.3) does not satisfy the strong section condition. However one can do a coordinate redefinition and set simply $(y_\alpha^1)_i = (y_\beta^1)_i + (\tilde{a}_{\alpha\beta}^1)_i$ but this is less attractive in calculations.

To find the patching conditions of the tangent bundle consider the vector field $X_\alpha = A_\alpha^i \frac{\partial}{\partial x_\alpha^i} + B_{\alpha,i} \frac{\partial}{\partial (y_\alpha)_i}$ and demanding $X_\alpha = X_\beta$, we find that

$$\begin{aligned} A_\alpha^i &= \frac{\partial x_\alpha^i}{\partial x_\beta^j} A_\beta^j, \\ (B_\alpha)_i &= \frac{\partial x_\beta^k}{\partial x_\alpha^i} (B_\beta)_k + \frac{\partial x_\alpha^m}{\partial x_\beta^p} \frac{\partial^2 x_\beta^k}{\partial x_\alpha^m \partial x_\alpha^i} (y_\beta)_k A_\beta^p + \frac{\partial (\tilde{a}_{\alpha\beta}^1)_i}{\partial x_\beta^k} A_\beta^k. \end{aligned} \quad (\text{A.5})$$

It is clear from this that $T\tilde{M}_{\omega^3}$ is an extension of π^*TM with respect to the cotangent bundle $\pi^*\Lambda^1 M$ of M , ie

$$0 \rightarrow \pi^*\Lambda^1 M \rightarrow T\tilde{M}_{\omega^3} \rightarrow \pi^*TM \rightarrow 0. \quad (\text{A.6})$$

In particular, $\pi^*\Lambda^1 M$ is a subbundle of $T\tilde{M}_{\omega^3}$. This is reminiscent of generalized geometry where the bundle E over M which is an extension of TM is now replaced with $T\tilde{M}_{\omega^3}$.

Furthermore observe that the pairing of TM and $\Lambda^1 M$ is naturally extended to $T\tilde{M}_{\omega^3}$ as the transition functions of $T\tilde{M}_{\omega^3}$ preserve it. As a result, one can define an $O(n, n)$ structure on $T\tilde{M}_{\omega^3}$ as expected from string theory considerations which also arises in the context of generalized geometry. But of course this $O(n, n)$ is related to the modified transition functions rather than those associated to the original patching conditions $a_{\alpha\beta}^1$ of the 2-form gauge potential.

In the more general case of \tilde{M}_{ω^k} , one finds that

$$0 \rightarrow \pi^*\Lambda^{k-2} M \rightarrow T\tilde{M}_{\omega^k} \rightarrow \pi^*TM \rightarrow 0. \quad (\text{A.7})$$

A.2 Testing for other properties

One of the properties of the construction of C-folds for 2-forms ω^2 is that when one pulls back the 2-form on the C-fold, ω^2 becomes exact. It is not expected that this property holds on \tilde{M}_{ω^k} because by construction the fibres have trivial topology, ie by construction \tilde{M}_{ω^k} is contractible to M . Nevertheless, it is instructive to see what the result is. In analogy with the 2-form case, we take the exterior derivative of second transition function in (5.10) to find

$$dy_\alpha^{k-2} - dy_\beta^{k-2} = C_\alpha^{k-1} - C_\beta^{k-1} + \sum_\gamma d\rho_\gamma \wedge da_{\alpha\beta\gamma}^{k-3}. \quad (\text{A.8})$$

Using the properties of the partition of unity and the patching conditions of ω^k , this can be rewritten as

$$dy_\alpha^{k-2} - C_\alpha^{k-1} - \sum_\gamma d\rho_\gamma \wedge a_{\gamma\alpha}^{k-2} = dy_\beta^{k-2} - C_\beta^{k-1} - \sum_\gamma d\rho_\gamma \wedge a_{\gamma\beta}^{k-2}. \quad (\text{A.9})$$

It is clear from the above equation that ω^k cannot be written as the exterior derivative of a $(k-1)$ -form on \tilde{M}_{ω^k} . For the latter, one would have expected that $dy_\alpha^{k-2} - C_\alpha^{k-1}$

should have patched globally as a $(k-1)$ -form on \tilde{M}_{ω^k} . But as we have shown this is not the case and the $(k-1)$ -form that patches globally receives a correction that depends on the derivative of the functions ρ_α which appear in the partition of unity. This correction is like a source term. Assuming that the partition of unity functions have compact support and that the good cover is very fine, ρ_α resemble delta-functions. Therefore the source term is like the derivative of a delta function. Note that in the construction of some exceptional field theories additional field are required in the form of tensor hierarchies, see [13, 14] which is reminiscent to these additional terms.

A.3 \tilde{M} for M-theory

It would be of interest in view of applications in strings and M-theory to generalize the construction of the previous appendix from single k -forms, ω^k , to differential algebras \mathcal{A} . We shall not give a general treatment of this. Instead we shall focus on the differential algebra of M-theory generated by the 4-form field strength F and its dual G , where now

$$\mathcal{A}: \quad dF = 0, \quad dG = F \wedge F, \quad (\text{A.10})$$

and G is treated as an independent field.

Suppose that M now has a good cover, the above equations can be solved at each open set of the cover as

$$F_\alpha = dC_\alpha^3, \quad G_\alpha = dC_\alpha^6 + C_\alpha^3 \wedge F_\alpha \quad (\text{A.11})$$

Next on double overlaps, we have

$$\delta C_{\alpha\beta}^3 = da_{\alpha\beta}^2, \quad \delta C_{\alpha\beta}^6 = db_{\alpha\beta}^5 - a_{\alpha\beta}^2 \wedge F \quad (\text{A.12})$$

and at triple overlaps

$$\delta a_{\alpha\beta\gamma}^2 = da_{\alpha\beta\gamma}^1, \quad \delta b_{\alpha\beta\gamma}^5 = db_{\alpha\beta\gamma}^4 + a_{\alpha\beta\gamma}^1 \wedge F \quad (\text{A.13})$$

Next to construct \tilde{M} , we introduce coordinates (x, y^2, w^5) and introduce the transition functions

$$\begin{aligned} x_\alpha &= f_{\alpha\beta}(x_\beta), \quad y_\alpha^2 - y_\beta^2 = a_{\alpha\beta}^2 - \sum_\gamma \rho_\gamma da_{\gamma\alpha\beta}^1, \\ w_\alpha^5 - w_\beta^5 &= b_{\alpha\beta}^5 - \sum_\gamma \rho_\gamma (a_{\gamma\alpha\beta}^1 \wedge F + db_{\gamma\alpha\beta}^4). \end{aligned} \quad (\text{A.14})$$

After performing a computation similar to that we have explained in previous case, the transition functions are consistent at triple overlaps. This proves that one can define a manifold $\tilde{M}_\mathcal{A}$ for the M-theory differential algebra \mathcal{A} in (A.10).

Most of the properties of the \tilde{M}_{ω^k} spaces constructed for single k -forms can be generalized to this case. First there is a projection from $\tilde{M}_\mathcal{A}$ onto M , and so $\tilde{M}_\mathcal{A}$ can be thought as a bundle over M . The tangent space $T\tilde{M}_\mathcal{A}$ is an extension of TM with respect to the space of 2- and 5-forms on M , ie

$$0 \rightarrow \pi^* \Lambda^2 M \oplus \pi^* \Lambda^5 M \rightarrow T\tilde{M}_\mathcal{A} \rightarrow \pi^* TM \rightarrow 0 \quad (\text{A.15})$$

As this construction is intended as an application to M-theory, M is 11-dimensional. However a similar construction can be applied to compactifications.

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